Int. J. Solids Structures Vol. 21, No. 10, pp. 1035–1045, 1985
Printed in Great Britain.

0020-7683/85 $0.00+.00
Pergamon Press Ltd.

MATERIAL CONSERVATION LAWS IN HIGHER-ORDER SHELL THEORIES

R. KIENZLER† and A. GOLEBIEWSKA-HERRMANN‡
Division of Applied Mechanics, Department of Mechanical Engineering, Stanford University, Stanford, CA 94305, U.S.A.

(*Received for publication* 5 *March* 1985)

Abstract—Within the framework of higher-order shell theories, a variational principle is established. The variation of the action integral includes the contributions due to changes of the field, boundaries and external loads. Based on the derived expression, a general discussion of possible material conservation laws is given.

1. INTRODUCTION

The conservation laws of material continua have received considerable attention during recent years[1–4]. This is probably due to the close relation of those new conservation laws to the path-independent integrals broadly used in fracture mechanics[5–8]. The material conservation laws express the invariance of the Lagrangian (or the action integral) under certain class of admissible material transformations[9].

Path-independent integrals in shell theory have been considered by several authors[10–12]. Let us notice that the investigations in [1–9] were concerned with infinite elastic continua. Shells are finite bodies, so the boundary conditions have to be included explicitly in any consideration. Roughly speaking, shell theory attempts to describe the real three-dimensional state of deformation (and stress) in terms of the deformed and undeformed configurations of the (curved) middle surface. The internal geometry, therefore, imposes the necessity of using curvilinear coordinates and, related to them, the special features of Riemannian geometry. For these reasons we have to establish the most general form of the variation principle for shells. From this principle the conditions for possible conservation laws can be derived.

Berger and Radenkovic[10] did not place any restrictions on the geometry of the shell middle surface. Based on the results of our study we conclude that their integrals are not path-independent in general.

Another difficulty, not related directly to the previous remarks, arises from the approximate character of shell theories. Since no "exact" theory is available, various features that characterize a consistent shell theory have been introduced to estimate the possible errors of the results and to indicate the range of applications of a given theory[13–16]. In order to meet these "quality criteria", *a priori* assumptions (like the classical Kirchhoff hypothesis) have to be used in establishing a first-order shell theory.

Nicholson and Simmonds[12] showed—in the context of shallow shell theory—that Sander's energy-release rate integral is path-independent for *all* mid-surface geometries. Lo[11] examined path-independent integrals for cylindrical shells and shells of revolution. Using the "best" first-order linear shell theory[13, 17] he expects that path-independent integrals do not exist in general for shells except those which enjoy a high degree of symmetry.

Therefore, it seems to be worthwhile to study conservation laws in the context of higher-order theories. To arrive at a shell theory of desired accuracy, special assumptions may be introduced later on in order to be sure that all effects and terms present is this degree of approximation are included.

† Visiting scholar, presently at Fraunhofer-Institut für Werkstoffmechanik, Wöhlerstr. 11, D-7800 Freiburg, FR Germany.
‡ Former Acting Associate Professor for Stanford University, now deceased.

Sets of equations of higher-order shell theories have been given by various authors[15, 16, 18, 19]. The following investigation is based on a general l-th–order shell theory.

2. GENERAL BACKGROUND

The middle surface in shell theory plays a very special role. Its significance can be compared with the role the elastic axis (centroidal locus of the cross-section) plays in beam theory. As independent variables for the shell description we choose the Gaussian coordinates ξ^α ($\alpha = 1, 2$) of the middle surface and the thickness parameter ζ as the third coordinate, perpendicular to the middle surface. All fields and equations governing their behavior are then expressed in terms of these variables and derivatives with respect to them.

The main goal of the shell theory is to find the approximate, but consistent, representation for the governing fields and field equations in terms of coordinates related to the middle surface only.

One way to approach this task (presented in [19]) consists in expanding all kinematic quantities in a power series with respect to ζ (assuming that thickness of a shell is much smaller than other surface dimensions). The local kinematic equations and constitutive laws are satisfied by requiring that the coefficients at each power ζ^n are equal on both sides of the initial relations. The global field equations are derived by integration with respect to ζ over the shell thickness. As a result, instead of using stresses τ^{ij} we deal with stress moments or stress resultants ${}^l m^{ij}$ which are weighted averages or l-th moments of the stress distribution in ζ direction.

The details of the above described procedure can be found in [15, 16, 18–20]. Therefore, here we restrict ourselves to a short account of the theory necessary for further development of conservation laws.

As usual, the range of Latin indices is 1, 2, 3 and that of Greek indices is 1 and 2. A vertical bar followed by a Greek index signifies covariant differentiation on the surface with respect to the indicated surface coordinate and a comma corresponds to partial differentiation. Let us consider a shell element (Fig. 1). The position vector **R**

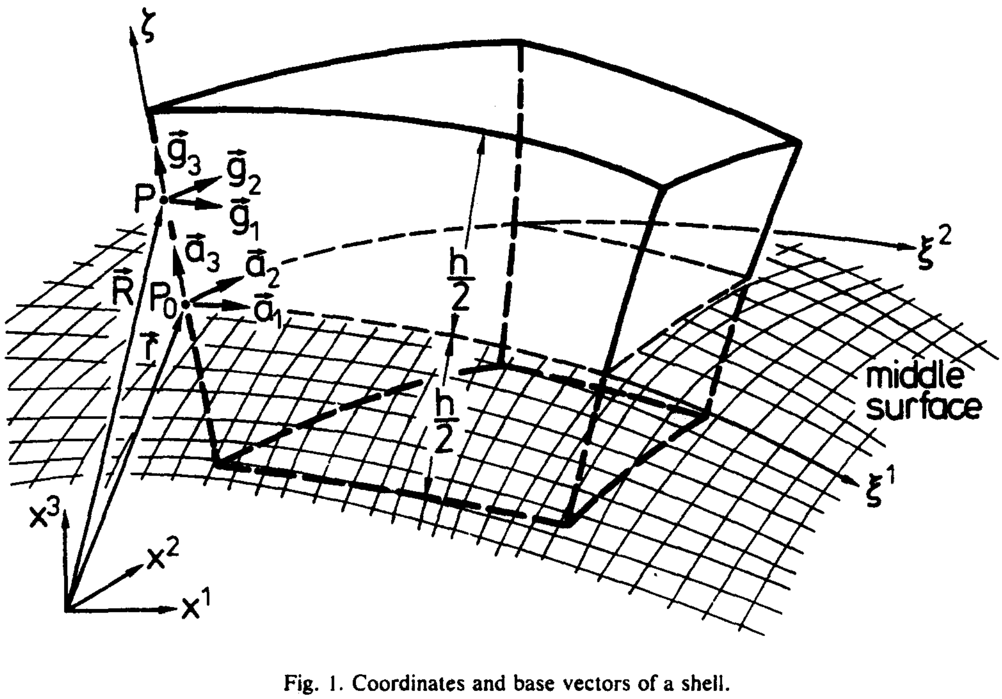

Fig. 1. Coordinates and base vectors of a shell.

of an arbitrary point P of the shell can be decomposed into a sum of two vectors: the position vector $\mathbf{r}$ of the corresponding point P_0 on the middle surfce and the vector $\mathbf{P_0P}$ along the coordinate line ζ (with unit vector $\mathbf{a}_3$ perpendicular to the middle surface):

$$\mathbf{R}(\xi^\alpha, \zeta) = \mathbf{r}(\xi^\alpha) + \zeta \mathbf{a}_3(\xi^\alpha). \tag{1}$$

The shifter μ^j_k is defined as

$$\mu^j_k = \delta^j_k - \zeta\, \delta^j_\alpha\, \delta^\beta_k\, b^\alpha_\beta, \tag{2}$$

where δ^j_k is the Kronecher delta and b^α_β is the (mixed) curvature tensor of the middle surface. The shifter connects the base vectors $\mathbf{g}_j$ at the arbitrary point P of the shell ($\mathbf{g}_j = \mathbf{R}_{,j}$) with base vectors $\mathbf{a}_\alpha$ at P_0 on the middle surface ($\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$)

$$\mathbf{g}_k = \mu^j_k \mathbf{a}_j. \tag{3a}$$

The displacement $\mathbf{u}$ can be written in terms of spatial components $u^{(S)i}$ such that $\mathbf{u} = u^{(S)}_i\, \mathbf{g}^i$ or surface components $u^{(A)i}$; $\mathbf{u} = u^{(A)}_i\, \mathbf{a}^i$. Again, the shifter relates both, spatial and surface representations of $\mathbf{u}$:

$$u^{(S)}_k = \mu^j_k u^{(A)}_j. \tag{3b}$$

The relation between the infinitesmal volume element $\mathrm{d}V$ of the shell and the infinitesimal element of the middle surface $\mathrm{d}A$ is given by [14]

$$\mathrm{d}V = \mu\, \mathrm{d}A\, \mathrm{d}\zeta, \tag{4}$$

where μ is the determinant of the shifter.

Generalized stress resultants ${}^l m^{ij}$ (= l-th moments of the stress distribution in the thickness direction) are defined by

$${}^l m^{ij} = \int_{-h/2}^{+h/2} \zeta^l \mu\, \tau^{ik}\, \mu^j_k\, \mathrm{d}\zeta, \qquad l = 0, 1, 2, \ldots. \tag{5}$$

The index l appearing on the upper left-hand side of the generic letter m is related to the order of the moment only, as obvious from the above formula, and has nothing to do with tensor indices appearing on the right-hand side of m. Some of these moments have a simple physical interpretation; the classical tensor of inplane forces (normal and shear) is given by ${}^0 m^{\alpha\beta}$ while ${}^1 m^{\alpha\beta}$, the moment's tensor, is related to bending and twisting. The components ${}^l m^{33}$ result from normal shear in the ζ direction.

As mentioned above, we expand the components of the displacement vector $u^{(A)}_j$ in a power series with respect to the thickness coordinate ζ,

$$u^{(A)}_j = {}^0 u_j + {}^1 u_j \zeta + {}^2 u_j \zeta^2 + \cdots = \sum_{l=0}^{\infty} {}^l u_j \zeta^l. \tag{6}$$

The first terms ${}^0 u_j$ correspond to displacements, ${}^1 u_\alpha$ contributes to the rotations of the normal of the shell middle surface. ${}^1 u_3$ is related to the change of thickness; higher-order coefficients ${}^l u_j$ ($l > 1$) include effects of warping of the cross-section.

We introduce the moments ${}^l W$ of the strain density $\overline{W}$

$${}^l W = \int_{-h/2}^{+h/2} \zeta^l \mu \overline{W}\, \mathrm{d}\zeta. \tag{7}$$

Now we can define the strain resultants ${}^l \lambda_{ij}$ conjugate to the stress resultants ${}^l m^{ij}$

defined by (5) in the following manner[19]:

$$
\begin{aligned}
{}^l\lambda_{\beta\alpha} &= {}^l u_\alpha |_\beta - {}^l u_3 \, b_{\alpha\beta}, \\
{}^l\lambda_{\alpha 3} &= {}^l u_3 |_\alpha + {}^l u_\beta \, b_\alpha^\beta, \\
{}^l\lambda_{3j} &= {}^{l+1}u_j(l + 1).
\end{aligned}
\tag{8}
$$

(Like the displacement gradient the quantities ${}^l\lambda_{ij}$ are not invariant under superposed rigid body motions. Only symmetric invariant quantities, however, occur in the arguments of the constitutive response functions[19].)

We can check that ${}^l m^{ij}$ and ${}^l\lambda_{ij}$ satisfy the following constitutive equations:

$$
\frac{\partial^l W}{\partial^n \lambda_{ij}} = {}^{l+n}m^{ij}. \tag{9}
$$

In a similar way we can introduce moments ${}^lV$ of the potential of external forces $\overline{V}$ (per unit volume). They are related to the force resultants ${}^l p^j$ through the following relations:

$$
\frac{\partial^l V}{\partial^n u_j} = -\, {}^{l+n}p^j. \tag{10}
$$

The external shell forces ${}^l p^j$ result from body forces $\bar{p}^j$ and prescribed stresses $\overset{*}{\tau}{}^{3k}$ acting on the upper and lower shell faces and are defined by

$$
{}^l p^j = \int_{-h/2}^{+h/2} \zeta^l \mu \, \bar{p}^k \, \mu_k^j \, d\zeta + [\zeta^l \mu \, \overset{*}{\tau}{}^{3k} \, \mu_k^j]_{-h/2}^{+h/2} \tag{11}
$$

Therefore ${}^lV$ has to be defined as

$$
{}^lV = \int_{-h/2}^{-h/2} \zeta^l \mu \, \overline{V} \, d\zeta - \left[\zeta^l \mu \, \overset{*}{\tau}{}^{3k} \, \mu_k^j \sum_{n=0}^{\infty} {}^n u_j \, \zeta^n \right]_{-h/2}^{+h/2}, \tag{12}
$$

where we took into account formulas (3) and (6).

We will illustrate the procedure described above in the next section; namely, we will derive the equilibrium equations and boundary conditions from the classical virtual work theorem.

Without going into details, it should be mentioned that from a mathematical point of view, the procedure presented above is based on the Weierstrass approximation theorem and convergence theorems proven by Sensenig[21].

In practice we limit ourselves to only the first few terms in the expansion. The first-order approximation is the simplest one (aside from the membrane theory); however, as mentioned in the introduction, additional assumptions and neglections are necessary to obtain a consistent first-order shell theory[16].

It turns out that an appropriate second-order approximation leads to consistent shell equations[19], in particular there are no contradictions arising from the eqns (9) and (10), i.e.

$$
\frac{\partial^l W}{\partial^n \lambda_{ij}} = \begin{cases} {}^{l+n}m^{ij} & \text{for } l,\ n,\ l + n \le 2; \\ 0 & \text{otherwise.} \end{cases} \tag{9'}
$$

Let us observe that all features of a consistent shell theory set up during past years[13–16, 18] are met, in particular, fulfillment of the static geometric analogy: no additional assumptions or corrections of the equations are necessary. It has been demonstrated[19(b)] that this theory reduces to accepted engineering beam theories.

3. THE CLASSICAL VIRTUAL WORK THEOREM: EQUILIBRIUM EQUATIONS AND BOUNDARY CONDITIONS

The stress resultants ${}^l m^{ij}$ have to satisfy the equilibrium equations and boundary conditions. In order to derive the equations for ${}^l m^{ij}$ we multiply the governing three-dimensional equations for a shell element by ζ^l and then integrate with respect to ζ over the thickness of the shell[19].

The alternate way, which we will follow here, is based on the use of the virtual work theorem for the shell theory. As a first step of this procedure we will introduce the moments ${}^l\Pi$ of the total energy Π of the shell.

The total energy Π of an elastic body occupying the region V in the static case consists of the sum of strain energy and potential of body forces diminished by the work done by the tractions on the boundary ∂V

$$\Pi = \int_V (\overline{W} + \overline{V})\,\mathrm{d}V - \int_{\partial V} \tau^{ji} n_j u_i \,\mathrm{d}\partial V. \tag{13}$$

If we apply the Gauss theorem to the last term, the total energy Π can be represented as the volume integral. Then we define moments ${}^l\Pi$ in the following way

$${}^l\Pi = \int_V \zeta^l[\overline{W} + \overline{V} - (\tau^{ji}u_i)\,|_j]\,\mathrm{d}V.$$

Integration by parts leads to another form of Π, namely

$${}^l\Pi = \int_V [\zeta^l(\overline{W} + \overline{V}) + l\zeta^{l-1}\tau^{3i}u_i]\,\mathrm{d}V - \int_{\partial V} \zeta^l\tau^{ji}u_i n_j \,\mathrm{d}\partial V.$$

Consider now a shell element bounded by the surfaces $\mathrm{d}A^+$ and $\mathrm{d}A^-$ ($\zeta = \pm h/2$) and a cross-sectional surface $\mathrm{d}S$ (unit outward normal $n_\alpha^{(S)}$) perpendicular to the middle surface (Fig. 2). The intersection between the cross-section and the middle surface is denoted by a smooth closed curve c (arc length l, unit outward normal $n_\alpha^{(A)} = n_\alpha$). According to [14] it is

$$n_\alpha^{(S)}\,\mathrm{d}S = \mu n_\alpha \,\mathrm{d}l\,\mathrm{d}\zeta, \qquad n_3^{(S)}\,\mathrm{d}A^{\pm} = \mu\,\mathrm{d}A\,|_{\pm h/2},$$

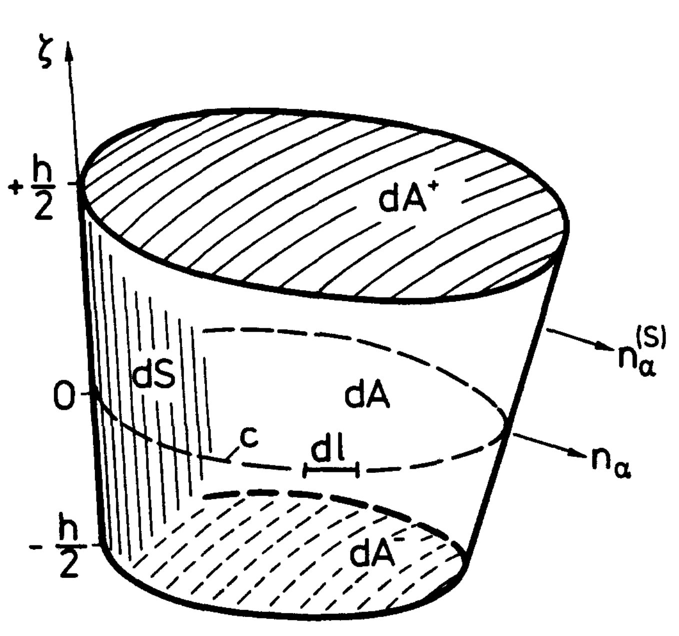

Fig. 2. Shell element with bounding surfaces.

and with eqns (3)–(7) and (12) we get finally

$$^{l}\Pi = \int_A \left({}^{l}W + {}^{l}V + l \sum_{n=0}^{\infty} {}^{l+n-1}m^{3k}\, {}^{n}u_k \right) \mathrm{d}A - \oint_c \sum_{n=0}^{\infty} \zeta^{l+n} \mu \tau^{\alpha i} \mu_i^k\, {}^{n}u_k n_\alpha\, \mathrm{d}l. \quad (14)$$

The integrand of the surface integral represents the l-th moment ${}^{l}L$ of the Lagrangian density $\overline{L}$ (with negative sign):

$$- {}^{l}L({}^{n}u_j, {}^{n}u_j|_\alpha, \xi^\alpha) = {}^{l}W + {}^{l}V + l \sum_{n=0}^{\infty} {}^{l+n-1}m^{3k}\, {}^{n}u_k. \quad (15)$$

Thus the integral itself is the l-th moment ${}^{l}A$ of the (negative) action A. The contour integral describes the work done by the given loads ${}^{l+n}p^{*k}$ acting on the cross-sections. With these observations we can simplify the form of eqn (14) to

$$^{l}\Pi = -{}^{l}A - \oint_c \sum_{n=0}^{\infty} {}^{l+n}p^{*k}\, {}^{n}u_k\, \mathrm{d}l. \quad (14')$$

The equilibrium equations can be directly derived from ${}^{l}L$ as Euler–Lagrange equations, i.e.

$$\left. \frac{\partial^l L}{\partial^n u_j|_\alpha} \right|_\alpha - \frac{\partial^l L}{\partial^n u_j} = 0.$$

As mentioned before, we need in addition the boundary conditions for ${}^{l}m^{ij}$. That is why we will go through the variational procedure for ${}^{l}\Pi$ involving not only ${}^{l}A$ but the work term as well. We will see that this will provide us with the equilibrium equations and the boundary conditions from the same variational principle. We compare the "neighboring" states of total energy (14) assuming that the independent variables ξ^α do not change. The requirement of the stationary of the functional ${}^{l}\Pi$ gives

$$0 = \bar{\delta}{}^{l}\Pi = \int_A \sum_{n=0}^{\infty} \left(\frac{\partial^l W}{\partial^n \lambda_{ij}} \bar{\delta}\, {}^{n}\lambda_{ij} + \frac{\partial^l V}{\partial^n u_j} \bar{\delta}\, {}^{n}u_j + l\, {}^{l+n-1}m^{3k}\, \bar{\delta}\, {}^{n}u_k \right) \mathrm{d}A - \oint_c \sum_{n=0}^{\infty} {}^{l+n}p^{*k}\, \bar{\delta}\, {}^{n}u_k\, \mathrm{d}l. \quad (16)$$

The symbol $\bar{\delta}$ denotes a variation when the boundaries are fixed, whereas δ will be used for more general variations, with varying boundaries, to be studied in the next section.

Using eqns (8)–(10) and integrating by parts $\bar{\delta}{}^{l}\Pi$ can be rewritten as

$$\begin{aligned} \bar{\delta}{}^{l}\Pi = \int_A \sum_{n=0}^{\infty} \{ & - [{}^{l+n}m^{\beta\alpha}|_\beta - b_\beta^\alpha\, {}^{l+n}m^{\beta 3} - (l+n)\, {}^{l+n-1}m^{3\alpha} + {}^{l+n}p^\alpha]\, \bar{\delta}\, {}^{n}u_\alpha \\ & - [{}^{l+n}m^{\beta 3}|_\beta + b_{\alpha\beta}\, {}^{l+n}m^{\beta\alpha} - (l+n)\, {}^{l+n-1}m^{33} + {}^{l+n}p^3]\, \bar{\delta}\, {}^{n}u_3 \\ & + \underline{({}^{l+n}m^{\beta j}\, \bar{\delta}\, {}^{n}u_j)|_\beta} \} \,\mathrm{d}A - \oint_c \sum_{n=0}^{\infty} {}^{l+n}p^{*j}\, \bar{\delta}\, {}^{n}u_j \,\mathrm{d}l. \end{aligned} \quad (17)$$

The underlined term in the surface integral can be transformed into a line integral; thus requiring $\bar{\delta}\, {}^{l}\Pi$ to vanish leads to the well-known equilibrium equations and boundary conditions

In V:

$$
\begin{aligned}
{}^{l+n}m^{\beta\alpha}|_\beta - b^\alpha_\beta\, {}^{l+n}m^{\beta 3} - (l+n)\, {}^{l+n-1}m^{3\alpha} + {}^{l+n}p^\alpha &= 0, \\
{}^{l+n}m^{\beta 3}|_\beta + b_{\alpha\beta}\, {}^{l+n}m^{\beta\alpha} - (l+n)\, {}^{l+n-1}m^{33} + {}^{l+n}p^3 &= 0.
\end{aligned} \tag{18a}
$$

In ∂V:

$$
({}^{l+n}m^{\beta j}\, n_\beta - {}^{l+n}p^{*j})\, \bar{\delta}\, {}^n u_j = 0. \tag{18b}
$$

The relations (18a) coincide with Euler–Lagrange eqns (16); but, at the same time, we established the boundary conditions, which can not usually be obtained from a stationary action integral. Equations following from the moment of momentum principle are omitted here, since they are not necessary to the following.

4. VARIATIONAL PRINCIPLE WITH VARYING BOUNDARIES

Classical variational principles have been used to derive equilibrium equations. It is impossible, however, to obtain, at the same time, material conservation laws because the class of variations is too restricted. In addition to "shape" variations $\bar{\delta}$ of the fields we have to admit the variations of the independent variables ξ^α contributing to the "convective variations" of the fields. This leads to a variational principle with varying boundaries[9, 22].

Let us consider a neighboring state of the action moment ${}^lA$:

$$
{}^lA^* = \int_A {}^lL^*\, ({}^n u_j^*, {}^n u_j^*|_\alpha; \xi^{*\alpha})\, dA^*. \tag{19}
$$

Let $\delta\xi^\alpha$ denote the variation of ξ^α, i.e.

$$
\delta\xi^\alpha = \xi^{*\alpha} - \xi^\alpha.
$$

Then, proceeding similarly as in [2], we get the following expressions for the (total) variations of ${}^n u_j$ and ${}^lL$:

$$
\begin{aligned}
\delta\, {}^n u_j &= \bar{\delta}\, {}^n u_j + {}^n u_j|_\alpha\, \delta\xi^\alpha + \mathcal{O}(\delta^2), \\
\delta\, {}^l L &= \bar{\delta}\, {}^l L + {}^l L|_\alpha\, \delta\xi^\alpha + \mathcal{O}(\delta^2).
\end{aligned} \tag{20}
$$

The transformed area dA^* is related to dA as follows:

$$
dA^* = dA[1 + (\delta\xi^\alpha)|_\alpha] + \mathcal{O}(\delta^2).
$$

The variation $\bar{\delta}\,{}^lL$ is equal to the underlined term in (17), since we omit the line integral in (14′) and the equations of equilibrium (18a) are supposed to be valid

$$
\bar{\delta}\,{}^lA = \int_A \bar{\delta}\,{}^lL\, dA = -\int_A \sum_{n=0}^{\infty} ({}^{l+n}m^{\beta j}\, \bar{\delta}\, {}^n u_j)|_\beta\, dA.
$$

Using all the above expressions for $\delta\,{}^l u_j$, $\delta\,{}^lL$, dA^* and $\bar{\delta}\,{}^lL$ we can rewrite (19) to derive the variation of ${}^lA$:

$$
\delta\,{}^lA = {}^lA^* - {}^lA = \int_A \left[\left({}^lL\delta^\beta_\alpha + \sum_{n=0}^{\infty} {}^{l+n}m^{\beta j}\, {}^n u_j \Big|_\alpha \right) \delta\xi^\alpha - \sum_{n=0}^{\infty} {}^{l+n}m^{\beta j}\, \delta\, {}^n u_j \right] \Bigg|_\beta dA + \mathcal{O}(\delta^2). \tag{21}
$$

The details of the above derivations and further references can be found in [2, 9, 22].

We are interested in infinitesimal transformations leaving the action integral A unchanged (invariant transformations of A), i.e. we require that $\delta' A = 0$. This can be expressed as the condition of vanishing of the line integral

$$\oint_c \left[\left({}^{l}L\delta^{\beta}_{\alpha} + \sum_{n=0}^{\infty} {}^{l+n}m^{\beta j}\, {}^{n}u_j \Big|_{\alpha} \right) \delta\xi^{\alpha} - \sum_{n=0}^{\infty} {}^{l+n}m^{\beta j}\, \delta\, {}^{n}u_j \right] n_{\beta}\, dl = 0. \quad (22)$$

We obtained it from (21) by Gauss theorem.

As a special case we consider first $\delta\xi^{\alpha} = 0$ —no change of the independent variables. Then the total variation of ${}^{l}u_j$ as given by eqns (20) reduces to $\bar{\delta}{}^{l}u_j$, i.e. $\delta{}^{l}u_i = \bar{\delta}{}^{l}u_i$. If in addition there are no body forces, i.e. all ${}^{l}p^{j} = 0$ and $\bar{\delta}{}^{l}u_i$ represent rigid body motions (infinitesimal), then eqn (22) leads to the rather known trivial statement that the forces applied on the boundaries have to be self-equilibrated.

Another class of transformations, namely

$$\delta{}^{l}u_j = 0, \qquad \delta\xi^{\alpha} \neq 0$$

is of special interest if we are concerned with material conservation laws of the system. Generalizing the Eshelby tensor[23] we introduce moments of the material momentum tensor ${}^{l}\psi^{\beta}_{\alpha}$:

$${}^{l}\psi^{\beta}_{\alpha} = {}^{l}L\delta^{\beta}_{\alpha} + \sum_{n=0}^{\infty} {}^{l+n}m^{\beta j}\, {}^{n}u_j|_{\alpha}. \quad (23)$$

The admissible transformations, leading to conservation laws, have to satisfy conditions following from (22)

$$\delta{}^{l}A = \oint_c {}^{l}\psi^{\beta}_{\alpha}\, \delta\xi^{\alpha}\, n_{\beta}\, dl = 0. \quad (24)$$

(We assumed $\delta{}^{l}u_j = 0$.) $\delta{}^{l}A$ can be rewritten, by means of the Gauss theorem, in the form

$$\delta{}^{l}A = \int_A \left\{ \left[{}^{l}L|_{\alpha} + \sum_{n=0}^{\infty} \left({}^{l+n}m^{\beta j}|_{\beta}\, {}^{n}u_j|_{\alpha} + {}^{l+n}m^{\beta j}\, {}^{n}u_j|_{\alpha\beta} \right) \right] \delta\xi^{\alpha} + {}^{l}\psi^{\beta}_{\alpha}\, \delta\xi^{\alpha}|_{\beta} \right\} dA. \quad (25)$$

Now we can calculate derivatives of ${}^{l}L$ [using relations (8), (9), (10) and (15)]:

$$\begin{aligned} {}^{l}L|_{\alpha} &= \frac{\partial{}^{l}L}{\partial\xi^{\alpha}}\Big|_{\exp} + \sum_{n=0}^{\infty} \left(\frac{\partial{}^{l}L}{\partial{}^{n}u_i}\, {}^{n}u_i|_{\alpha} + \frac{\partial{}^{l}L}{\partial{}^{n}u_{j|\beta}}\, {}^{n}u_j|_{\beta\alpha} \right) \\ &= \frac{\partial{}^{l}L}{\partial\xi^{\alpha}}\Big|_{\exp} - \sum_{n=0}^{\infty} \{ {}^{l+n}m^{\beta j}\, {}^{n}u_j|_{\beta\alpha} \\ &\quad + (b^{\gamma}_{\beta}\, {}^{l+n}m^{\beta 3} + (l+n)\, {}^{l+n-1}m^{3\gamma} - {}^{l+n}p^{\gamma})\, {}^{n}u_{\gamma}|_{\alpha} \\ &\quad + (-b_{\gamma\beta}\, {}^{l+n}m^{\beta\gamma} + (l+n)\, {}^{l+n-1}m^{33} - {}^{l+n}p^{3})\, {}^{n}u_3|_{\alpha} \}. \end{aligned}$$

From the above and the equilibrium eqns (18a) we get finally

$$\delta{}^{l}A = \int_A \left\{ \left[\frac{\partial{}^{l}L}{\partial\xi^{\alpha}}\Big|_{\exp} + \sum_{n=0}^{\infty} {}^{l+n}m^{\beta j} ({}^{n}u_j|_{\alpha\beta} - {}^{n}u_j|_{\beta\alpha}) \right] + {}^{l}\psi^{\beta}_{\alpha}\, \delta\xi^{\alpha}|_{\beta} \right\} dA. \quad (26)$$

We will use formula (26) for further discussion of possible conservation laws. In particular, we are interested in the possible contributions to the first form $[\partial^l L/\partial\xi^\alpha]_{\text{exp}}$. These can arise from various parameters involved in the descriptions like material properties and geometry of the shell itself. For a shell made of an isotropic material, ${}^lL$ has in general the following form:

$$ {}^lL = {}^lL(E, \nu, h, a_{\beta\gamma}, b^\gamma_\beta, {}^np^j; {}^nu_j, {}^nu_j|_\alpha), $$

where E and ν are elastic constants (Young's modulus, Poisson's ratio, respectively).

All the parameters up to the semicolon can contribute to the first term in (26). Therefore the conditions to be satisfied in order to ensure $[\partial^l L/\partial\xi^\alpha]_{\text{exp}} = 0$ are

$$ E, \nu, h = \text{const}, \tag{27a} $$

$$ {}^np^j|_\alpha = 0, \tag{27b} $$

$$ b^\gamma_\beta|_\alpha = 0. \tag{27c} $$

The condition $a_{\beta\gamma}|_\alpha = 0$ is satisfied automatically (Ricci's lemma). Next we will discuss the possible contributions from further terms in the integrand of (26). The form in parentheses in curved space is not necessarily zero because the differentiation is not commutative, i.e.

$$ {}^nu_\gamma|_{\alpha\beta} - {}^nu_\gamma|_{\beta\alpha} = R^\rho{}_{\gamma\alpha\beta}\,{}^nu_\rho, \tag{28} $$

where

$$ R^\rho{}_{\gamma\alpha\beta} = b_{\gamma\beta}b^\rho_\alpha - b_{\gamma\alpha}b^\rho_\beta \tag{29} $$

is the Riemann–Christoffel curvature tensor[14]. For the third component of ${}^n\mathbf{u}$, perpendicular to the middle surfaces, we can interchange the indices α and β, i.e.

$$ {}^nu_3|_{\alpha\beta} - {}^nu_3|_{\beta\alpha} = 0. $$

Assuming that E, ν and h are constants and (27b) is satisfied, we can see from (28) that δ^lA as given by (26) can be rewritten in the form convenient for further discussion

$$ \delta^lA = \int_A \left[\left(\frac{\partial^l L}{\partial b^\gamma_\beta} b^\gamma_\beta|_\alpha + R^\rho{}_{\gamma\alpha\beta} \sum_{n=0}^{\infty} {}^{l+n}m^{\beta\gamma}\,{}^nu_\rho\right)\delta\xi^\alpha + {}^l\psi^\beta_\alpha\,\delta\xi^\alpha|_\beta\right] dA = 0. \tag{30} $$

From the formula (30) it is easy to see the differences arising in establishing material conservation laws in shell theory and in general three-dimensional elasticity (in Euclidean space). Especially in the last case $R^\rho{}_{\gamma\alpha\beta}$ equals zero and, of course, $b^\beta_\gamma|_\alpha = 0$, and then the only contributions to the conservation laws are related to the moments of the material momentum ${}^l\psi^\beta_\alpha$. In addition to the requirement of homogeneous shell material (E, ν, h = const) the possibility of establishing material conservation laws in shell theory is reduced for the following reasons:

(i) (27c) is satisfied only for shells of constant curvature;
(ii) the components $R^\rho{}_{\gamma\alpha\beta}$ vanish only for developable surfaces;
(iii) the covariant derivatives of $\delta\xi^\alpha$ do not vanish for $\delta\xi^\alpha$ = const, except for circular cylindrical shells.

These restictions depend only on the shape of the shell middle surface and are therefore valid for all kinds of shell theories.

Lo[11] established conservation laws for shells based on the first-order shell theory[17]. The corresponding energy-momentum tensor can be derived from eqn (23) by introducing the assumptions and order-of-magnitude estimations commonly used in first order theory. Lo showed that conservation laws hold for cylindrical shells due to translational invariance and for shells of revolution due to translational invariance variance only in circumferential direction. These transformations, of course, lead to material conservation laws in higher-order shell theories, too.

In contrast to cylindrical shells it is possible to show that conservation laws hold for spherical shells not only due to translational invariance but also due to rotational invariance with respect to the normal of the middle surface.

5. CONCLUDING REMARKS

In the framework of a l-th–order shell theory the general condition for possible conservation laws has been derived. By contrast to the infinite elastic medium, it involves not only the material momentum tensor (or, in other words, the static part of Eshelby's energy-momentum tensor), but the boundary conditions and the geometric parameters of the shell. The latter is manifested by the presence of the curvature tensor of the middle surface, while the former by the resultants of the external load.

The comparison with conservation laws based on the first-order shell theory indicates that conditions obtained here are more general and the previous ones can be obtained from them by neglecting some terms (or effects) present in a higher-order theory. More over, from the expression of the material momentum tensor presented here, we arrive at material momentum tensor for shell theories of any order of approximation by introducing their assumptions and order-of-magnitude estimations.

By checking the restrictions imposed on the existence of path-independent integrals the following conclusions can be drawn:

Only for spherical shells material translations and rotations are admissible transformations leading to conservation laws. Circular cylindrical shells allow material translations in circumferential and longitudinal direction. Regarding shells of revolution, a material translation in circumferential direction is an admissible transformation, while a translation in meridional direction is not. Thus, by contrast to the infinite Euclidian space, the "interchange" of coordinates in deriving corresponding conservation laws is not possible.

Probably further transformations exist (e.g. due to self-similarity) leading to material conservation laws. Such admissible transformations have to be discovered in the future.

Because of the outlined difficulties resulting from the geometry on a curved surface and the approximate character of shell theories, conservation laws for shells have to be handled with special caution, i.e. we have to be sure that a conservation law established in a nth-order shell theory is consistent with that approximation.

Acknowledgements—The authors gratefully acknowledge the discussions with Prof. George Herrmann and his greatest patience in correcting our poor English. This research was supported by a fellowship of the Deutsche Forschungsgemeinschaft to Dr.-Ing. R. Kienzler and by the Department of Energy through Grant No. DOE 12040 to Stanford University.

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